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Infinite Products for Power Series

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An algorithm is introduced and shown to lead to a unique infinite product representation for a given formal power series $A(z)$ with $A(0)=1$. The infinite product is

$$\prod_{n=1}^{\infty} (1+b_n z^{r_n}),$$

where all $b_n \neq 0$, $r_n \in \mathbb{N}$, and $r_{n+1} > r_n$. The degree of approximation by the polynomial $(1+b_1 z^{r_1}) \cdots (1+b_n z^{r_n})$ is also considered. © 1989 Academic Press, Inc.

1. INTRODUCTION

Let $\mathscr{L} = F((z))$ denote the field of all *formal Laurent series*

$$A = A(z) = \sum_{n=\nu}^{\infty} c_n z^n$$

in an indeterminate z, and with coefficients c_n all lying in a given field F. Although the main case of importance here occurs when F is the field $\mathbb{C}$ of complex numbers, certain interest also attaches to other ground fields F, and all results below hold for arbitrary F.

If $c_\nu \neq 0$, we call $\nu = \nu(A)$ the *order* of A, and define the *norm* (or *valuation*) of A to be

$$\|A\| = 2^{-\nu(A)}.$$

Letting $\nu(0) = +\infty$, $\|0\| = 0$, one then has (cf. Jones and Thron [2, Chap. 5]):

$$(1.1) \qquad \nu(AB) = \nu(A) + \nu(B), \qquad \nu\left(\frac{A}{B}\right) = \nu(A) - \nu(B) \qquad \text{if} \quad B \neq 0,$$

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and

$$v(\alpha A+\beta B)\geqslant \min(v(A), v(B)) \qquad \text{for} \quad 0\neq\alpha, \quad 0\neq\beta\in F,$$

with *equality* when $v(A)\neq v(B)$; and

$$(1.2) \qquad \|A\|\geqslant 0, \quad \|A\|=0 \quad \text{iff} \quad A=0, \quad \|AB\|=\|A\|\,\|B\|,$$

$$\|\alpha A+\beta B\|\leqslant \max(\|A\|, \|B\|) \qquad \text{for} \quad 0\neq\alpha, \quad 0\neq\beta\in F,$$

with *equality* when $\|A\|\neq\|B\|$.

By (1.2), the above norm is non-archimedean and leads to an *ultrametric* distance function ρ on $\mathscr{L}$, with $\rho(A, B)=\|A-B\|$. It is then essentially "folk-lore" that $\mathscr{L}$ forms a *complete* metric space under ρ (cf. Jones and Thron [2, Chap. 5], and the similar theorem for formal power series rings in Zariski and Samuel [4, Chap. VII]). Our main attention below will be confined to the closed subspace $\mathscr{P}_1\subset\mathscr{L}$ consisting of all formal power series $1+\sum_{n=1}^{\infty} c_n z^n$, our main result being:

(1.3) THEOREM. *Every $A\in\mathscr{P}_1$ with $A\neq 1$ has a unique convergent product representation (relative to ρ) of the form*

$$A=\prod_{n=1}^{\infty}(1+b_n z^{r_n}),$$

where $0\neq b_n\in F$, $r_n\in\mathbb{N}$, and $r_{n+1}>r_n$. Further,

$$v(A-P_n)\geqslant r_{n+1}>n, \qquad \textit{where} \quad P_n=(1+b_1z^{r_1})\cdots(1+b_nz^{r_n}).$$

This theorem is quite analogous to one on *continued fraction* expansions of elements of $\mathscr{P}_1$, due to Leighton and Scott [3], of the form

$$A=1+\frac{b_1z^{r_1}}{1}\,\underset{+}{}\,\frac{b_2z^{r_2}}{1}\,\underset{+}{}\,\frac{b_3z^{r_3}}{1}\,\underset{+}{}\,\cdots,$$

later called *C-fractions.*

In the final section, a few remarks are made about ordinary convergence to complex analytic functions.

2. CONVERGENCE AND APPROXIMATION

If $1\neq A\in\mathscr{P}_1$, let $\hat{A}_1=A=1+b_1z^{r_1}A_1'$ where $A_1'\in\mathscr{P}_1$ and $r_1\in\mathbb{N}$, $0\neq b_1\in F$. If $\hat{A}_n=1+b_nz^{r_n}A_n'$ has already been defined, with $1\neq A_n'\in\mathscr{P}_1$, $0\neq b_n\in F$, then define

$$\hat{A}_{n+1} = (1 + b_n z^{r_n})^{-1} \hat{A}_n = (1 - b_n z^{r_n} + b_n^2 z^{2r_n} - \cdots)(1 + b_n z^{r_n} + \cdots)$$
$$= 1 + b_{n+1} z^{r_{n+1}} A'_{n+1},$$

where $A'_{n+1} \in \mathscr{P}_1$, $r_{n+1} > r_n$, and $0 \neq b_{n+1} \in F$, if $A'_n \neq 1$. If $A'_n = 1$, let $\hat{A}_{n+1} = 1$ and stop the algorithm. Then

$$A = \hat{A}_1 = (1 + b_1 z^{r_1}) \hat{A}_2 = \cdots = \hat{A}_{n+1} \prod_{i=1}^{n} (1 + b_i z^{r_i}).$$

If the procedure does not terminate with some $\hat{A}_{n+1} = 1$, then

$$v(\hat{A}_{n+1} - 1) = r_{n+1} \geqslant 1 + r_n \geqslant \cdots \geqslant n + 1 \to \infty \qquad \text{as} \quad n \to \infty.$$

Thus $\lim_{n\to\infty} \hat{A}_{n+1} = 1$, and then $A = \prod_{i=1}^{\infty} (1 + b_i z^{r_i})$, relative to the metric ρ on $\mathscr{L}$.

If $P_n = P_n(z) = (1 + b_1 z^{r_1}) \cdots (1 + b_n z^{r_n})$, we also have

$$A = (1 + b_{n+1} z^{r_{n+1}} A'_{n+1}) P_n,$$

and so $v(A - P_n) = v(b_{n+1} z^{r_{n+1}} A'_{n+1} P_n) = r_{n+1} > n$.

3. Uniqueness of Representation

(3.1) Lemma. *Any product $\prod_{n \geqslant 1} (1 + d_n z^{s_n})$ with $0 \neq d_n \in F$, $s_n \in \mathbb{N}$, and $s_{n+1} > s_n$, converges relative to ρ to an element $B \neq 1$ in $\mathscr{P}_1$ with $B = 1 + d_1 z^{s_1} B'$, $B' \in \mathscr{P}_1$.*

Proof. If $\mathbb{Q}_n = \prod_{n=1}^{N} (1 + d_n z^{s_n})$, then

$$\mathbb{Q}_{N+K} - \mathbb{Q}_N = \mathbb{Q}_N R_{N,K},$$

where

$$R_{N,K} \prod_{n=N+1}^{N+K} (1 + d_n z^{s_n}) - 1 = d_{N+1} z^{s_{N+1}} R'_{N,K}$$

for $R'_{N,K} \in \mathscr{P}_1$, since (s_N) is a strictly increasing sequence. Thus

$$v(\mathbb{Q}_{N+K} - \mathbb{Q}_N) = v(\mathbb{Q}_N) + v(R_{N,K}) = s_{N+1} > N \to \infty$$

as $N \to \infty$, independently of $K \geqslant 1$. Therefore, if the product is infinite, then $(\mathbb{Q}_N)$ forms a Cauchy sequence and converges relative to ρ to an element B of the complete metric space $\mathscr{L}$. Since the convergence is relative to ρ, $v(B - \mathbb{Q}_N) > s_1$ for N sufficiently large, and so $B = 1 + d_1 z^{s_1} B'$ for $B' \in \mathscr{P}_1$, because $\mathbb{Q}_N = 1 + R_{0,N} = 1 + d_1 z^{s_1} R'_{0,N}$.

Now suppose that $A=\prod_{n\geq 1}(1+b_nz^{r_n})=\prod_{n\geq 1}(1+d_nz^{s_n})$ where $0\neq b_n$, $0\neq d_n\in F$, $r_n, s_n\in\mathbb{N}$, and $r_{n+1}>r_n$, $s_{n+1}>s_n$. Then Lemma 3.1 implies that

$$A=1+b_1z^{r_1}A'=1+d_1z^{s_1}B' \qquad \text{for} \quad A', B'\in\mathscr{P}_1.$$

Hence $b_1=d_1$, $r_1=s_1$, and we obtain $\prod_{n\geq 2}(1+b_nz^{r_n})=\prod_{n\geq 2}(1+d_nz^{s_n})$. In the same way, $b_2=b_2$ and $r_2=s_2$, and thus we successively obtain $b_n=d_n$, $r_n=s_n$ for all n.

Remarks. It is interesting to note from the general form of the product for A, and from simple examples like

$$(1+z)(1+2z)=(1+3z)(1+2z^2)(1-6z^3)\cdots,$$

that "most" polynomials have *infinite* representations as in Theorem 1.3, the exceptions being those of the general type P_n of Theorem 1.3. For polynomials of type P_n, the above algorithm then provides an actual *method* for determining all the roots.

It is also notable that an entire function without any ordinary zeros, like

$$e^z=1+z+\frac{z^2}{2}+\frac{z^3}{6}+\cdots=(1+z)\left(1+\frac{z^2}{2}\right)\left(1-\frac{z^3}{3}\right)\left(1+\frac{3z^4}{8}\right)\left(1-\frac{z^5}{5}\right)\cdots,$$

and in general every element $A\in\mathscr{P}_1$, acquires a new set (usually infinite) of "pseudo-zeros" as the result of Theorem 1.3. For example, -1, $\pm\sqrt{-2}\geq 3^{1/3}$, $3^{1/3}(-\frac{1}{2}\pm\frac{1}{2}\sqrt{-3})$, ... for e^z. Only in rare cases would these pseudo-zeros be genuine zeros. Nevertheless, a study of the relationship between properties of the pseudo-zeros and properties of A or of its power series coefficients might perhaps prove interesting. As two little comments in this direction, we note:

(i) If $A=1+b_1z^{r_1}+b_2z^{r_2}+\cdots$, where $r_1<r_2$, $b_i\neq 0$, then $A=(1+b_1z^{r_1})(1+b_2z^{r_2})\cdots$, where, however, the subsequent factors of the product need *not* necessarily conform to this initial pattern.

(ii) For a simple rational function $(1-bz^r)^{-1}$, with $b\neq 0$, $r\in\mathbb{N}$, the pseudo-zeros are given explicitly by the Euler-type identity which occurs in the proof of Proposition 4.1 below.

4. Analytic Functions and Ordinary Convergence

In the important case when F is the field $\mathbb{C}$ of complex numbers, the main interest of $\mathscr{L}$ or $\mathscr{P}_1$ lies in their close connection with ordinary complex meromorphic or analytic functions at the origin. As in the analytic theory of continued fractions (cf. Jones and Thron [2, Chap. 5], and Leighton and Scott [3]), it would be interesting to investigate the ordinary (uniform, if possible) convergence of the earlier approximation polynomials $P_n(z)$ to an actual analytic function $A(z)$.

Unfortunately, even the well-developed theory of continued fractions seems to be only partially successful in this genuinely analytic direction, seeming to work best only for special classes of functions and fractions. In effect, much of the theory of continued fraction approximations still seems to rest largely on a formal "correspondence" theory analogous to Theorem 1.3. Thus the present type of approximation and representation should probably be viewed in a similar light.

We therefore conclude first by merely mentioning that Jones and Thron [1, 2] have established at least one general theorem on uniform convergence to meromorphic functions under suitable conditions, which in principle is just as applicable to product as to continued fraction approximations (cf. Theorem 4 of [1], or Theorem 5.11 of [2]).

Second, we note that there remains the possibly interesting problem of characterizing the types of products in Theorem 1.3 which correspond to *rational* functions $A \in F(z)$. In the analogous case of the C-fraction expansions of elements of $\mathscr{P}_1$, due to Leighton and Scott [3], rational functions are characterized by finite continued fractions. However, since finite expansions as in Theorem 1.3 yield polynomial functions, a characterization of non-polynomial rational as well as most polynomial functions would therefore have to involve *infinite* products, satisfying possibly certain stronger growth conditions on the exponents r_n. As a partial result in this direction we note:

(4.1) PROPOSITION. *Let $A \in \mathscr{P}_1$, $A \neq 1$, with unique product representation*

$$A = \prod_{n=1}^{\infty} (1 + b_n z^{r_n}).$$

If $b_{n+1} = b_n^2$ and $r_{n+1} = 2r_n$, for all n sufficiently large, then $A \in F(z)$.

Proof. This follows easily from the Euler-type identity

$$\prod_{n=1}^{\infty} (1 + y^{2^{n-1}}) = \frac{1}{1-y} \qquad (v(y) > 0).$$

(To establish this identity, note that the equation $1 + w = (1 - w^2)/(1 - w)$ $(w \neq 1)$ leads, as for real numbers, to

$$\prod_{n=1}^{N} (1 + y^{2^{n-1}}) = \frac{1-y^2}{1-y} \cdot \cdots \cdot \frac{1-y^{2^N}}{1-y^{2^{N-1}}} = \frac{1-y^{2^N}}{1-y}.$$

The identity then follows, since $v(y^k) = kv(y) \to \infty$ as $k \to \infty$, i.e. $y^k \to 0$ relative to ρ.

Note added in proof. When $F = \mathbb{C}$, some analytical properties of the above product and combinatorial properties of its coefficients have been discussed by J. F. Ritt (*Math. Z.* **32** (1930), 1–3) and by H. Gingold, H. W. Gould and M. E. Mays (*Utilitas Math.* **34** (1988), 143–161), respectively.

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